<u>Reminder: the normalized second order coherence function:</u>

$$\mathrm{g}^{(2)}(\boldsymbol{r_1},t_1,\boldsymbol{r_2},t_2)=\frac{\langle \hat{a}^\dagger\,(\boldsymbol{r_1},t_1)\hat{a}^\dagger\,(\boldsymbol{r_2},t_2)\hat{a}(\boldsymbol{r_2},t_2)\hat{a}(\boldsymbol{r_1},t_1)\rangle}{\langle \hat{a}^\dagger\,(\boldsymbol{r_1},t_1)\hat{a}(\boldsymbol{r_1},t_1)\rangle\langle \hat{a}^\dagger\,(\boldsymbol{r_2},t_2)\hat{a}(\boldsymbol{r_2},t_2)\rangle}$$

Naturally, this is called the 2nd-order quantum correlation function (Mandel&Wolf call it the 4th.. but they are the only ones).

<u>Classical correlation functions:</u>

See here the consequences of the normal ordering – if it was ordered differently, as is the case classically, where these are not operators but amplitudes, we would have (no "hats"):

$$\mathrm{g}_\mathrm{c}^{(2)}(\boldsymbol{r_1},t_1,\boldsymbol{r_2},t_2)=\frac{\langle a^\dagger\,(\boldsymbol{r_1},t_1)a^\dagger\,(\boldsymbol{r_2},t_2)a(\boldsymbol{r_2},t_2)a(\boldsymbol{r_1},t_1)\rangle}{\langle a^\dagger\,(\boldsymbol{r_1},t_1)a(\boldsymbol{r_1},t_1)\rangle\langle a^\dagger\,(\boldsymbol{r_2},t_2)a(\boldsymbol{r_2},t_2)\rangle}$$

$$=\frac{\langle a^\dagger\,(\boldsymbol{r_1},t_1)a(\boldsymbol{r_1},t_1)a^\dagger\,(\boldsymbol{r_2},t_2)a(\boldsymbol{r_2},t_2)\rangle}{\langle a^\dagger\,(\boldsymbol{r_1},t_1)a(\boldsymbol{r_1},t_1)\rangle\langle a^\dagger\,(\boldsymbol{r_2},t_2)a(\boldsymbol{r_2},t_2)\rangle}$$

$$=\frac{\langle I(\boldsymbol{r_1},t_1)I(\boldsymbol{r_2},t_2)\rangle}{\langle I(\boldsymbol{r_1},t_1)\rangle\langle I(\boldsymbol{r_2},t_2)\rangle}$$

i.e. intensity correlations.

Semiclassically they mean that the chance for arrival of a photon is proportional to the field^2=the intensity, which is what we learned in Modern Physics.

The similarity to the 1st order normalized correlation function made many (including the green book, Wikipedia and others) call it "**Second order Coherence**". However there is NO basis for that – 1 means nothing, and 0 is definitely not "ordinary" and not classical – it is completely non classical, and if possible, needs a lot of "coherence". So I (and Mandel&Wolf support me here..) will call it normalized 2nd order coherence function and only that.

The classical consequences of noise/fluctuations :

This expression, with Intensities not Operators obeys trivial inequalities, such as:

$$\langle \boldsymbol{I^2}\rangle \geq \langle \boldsymbol{I}\rangle^{\boldsymbol{2}}$$

Postive variance of course:

$$Var(\boldsymbol{I})=\langle \boldsymbol{I^2}\rangle-\langle \boldsymbol{I}\rangle^{\boldsymbol{2}}$$

THIS MEANS that the ONLY way in which we get equality is if $\boldsymbol{I}$ is constant – once there is even slight fluctuations/noise, then $\langle \boldsymbol{I^2}\rangle > \langle \boldsymbol{I}\rangle^{\boldsymbol{2}}$

, So

$$g_c^{(2)}(\boldsymbol{r_1} = \boldsymbol{r_2}, t_1 = t_2) \geq 1$$

And if the light is not completely constant we get more than 1.

And also there is the Cauchy-Schwartz inequality for random variables:

$$|\langle \boldsymbol{I_1 I_2} \rangle|^2 \leq \langle \boldsymbol{I_1}^2 \rangle \langle \boldsymbol{I_2}^2 \rangle$$

Which means that

$$g_c^{(2)}(\boldsymbol{r_1}, t_1, \boldsymbol{r_2}, t_2) \leq g_c^{(2)}(\boldsymbol{r_1} = \boldsymbol{r_2}, t_1 = t_2)$$

For example, if we take $\boldsymbol{r_1} = \boldsymbol{r_2}$**,** and look at stationary processes, then $g^{(2)}$ does not depend on t_1, t_2 specifically, but only on the separation between them $\tau = t_2 - t_1$,

And so we get:

$$g_c^{(2)}(\tau) = \frac{\langle I(t) I(t+\tau) \rangle}{\langle I(t) \rangle \langle I(t+\tau) \rangle}$$

$$= for\ stationary\ we\ can\ also\ change\ the\ dominator\ to\ get =$$

$$= \frac{\langle I(t) I(t+\tau) \rangle}{\langle I(t) \rangle^2}$$

But I don't like this normalization since experimentally things are never really stationary, and people tend to forget the original expression.

So we get

$$g_c^{(2)}(0) \geq 1$$

and

$$g_c^{(2)}(\tau) \leq g_c^{(2)}(0)$$

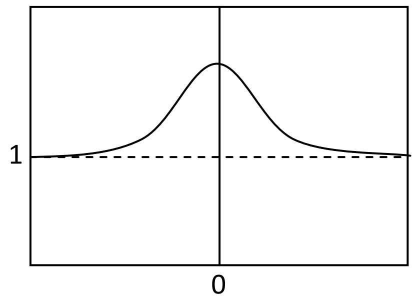
All this inequalities are called **Classical Inequalities**, because they were driven using classical assumptions. Any light source that violates these inequalities is inherently NON-CLASSICAL

Since it violates the semi-classical assumption that the probability for the arrival of a photon is simply related to the intensity of the light field (i.e. separability, no entanglement etc..).

Naturally, for stationary processes (that never end, and anyway for non-stationary you cannot draw g2 as a function of a single parameter) we can also expect that far enough we will have:

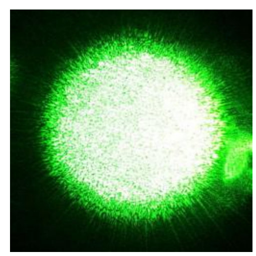
$$g_c^{(2)}(\infty) = 1$$

So we expect a graph that looks like that:



Which means nothing can be MORE correlated than the signal with itself, and that these correlations must decay to the value normalized to 1, and so g2(0)=1 means g2=1 EVERYWHERE, ZERO variance, i.e. classically – no fluctuations, no noise whatsoever.. constant source = indefinite coherence. Once there IS some noise/fluctuations, there will also be the coherence LENGTH/TIME – so for longer delays we expect no correlations between the intensities, and for shorter delays we expect g2>1. So we can show indeed that this "coherence length" is indeed THE coherence length. As will be shown in the tutorial, g2 of chaotic (=random phase) sources is simply 1+g1, WHICH EXPLAINS EVERYTHING.

In particular we get for chaotic light g2(0)=2, which makes sense in terms of amplitude vs. power of constructive interference.

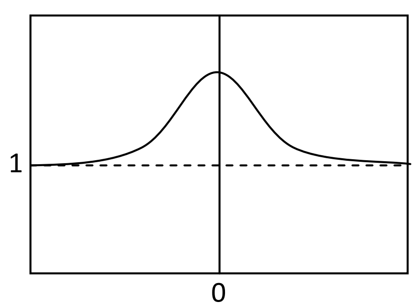


No go back to the speckle pattern – remember that the size of the speckle

Is 1/size of the source, and that naturally this is also the coherence area

(and in time the coherence time is 1/BW)

Based on these coherence lengths we saw the application of Stellar interferometry and spectroscopy. But there is something even easier – just auto-correlate the Intensities, and of course we'll get :



And this is the basis for HBT:

Hanbbury Brown and Twiss (HBT) :

R. Hanbury Brown and R. Q. Twiss (1956). "A Test of a New Type of Stellar Interferometer on Sirius". Nature 178 (4541): 1046–1048

E. Purcell (1956). "The Question of Correlation Between Photons in Coherent Light Rays". Nature 178 (4548): 1449–1450.

R. Hanbury Brown and R. Q. Twiss (1957). "Interferometry of the intensity fluctuations in light. I. Basic theory: the correlation between photons in coherent beams of radiation". Proc of the Royal Society of London A 242 (1230): 300–324.

R. Hanbury Brown and R. Q. Twiss (1958). "Interferometry of the intensity fluctuations in light. II. An experimental test of the theory for partially coherent light". Proc of the Royal Society of London A 243 (1234): 291–319

Fano, U. (1961). "Quantum theory of interference effects in the mixing of light from phase independent sources". American Journal of Physics 29 (8): 539.

B. L. Morgan and L. Mandel (1966). "Measurement of Photon Bunching in a Thermal Light Beam". Phys. Rev. Lett. 16 (22): 1012–1014.

Kimble, H.; Dagenais, M.; Mandel, L. (1977). "Photon antibunching in resonance fluorescence". Physical Review Letters 39 (11): 691.

P. Grangier, G. Roger, and A. Aspect (1986). "Experimental Evidence for a Photon Anticorrelation Effect on a Beam Splitter: A New Light on Single-Photon Interferences". Europhysics Letters 1 (4): 173–179.

Y. Bromberg, Y. Lahini, E. Small and Y. Silberberg (2010). "Hanbury Brown and Twiss Interferometry with Interacting Photons". Nature Photonics 4 (10): 721–726. Bibcode 2010NaPho...4..721B. doi:10.1038/nphoton.2010.195

The impressive thing here is the list of famous authors, beside HBT – Fano, Grangier, Mandel, Kimble, Aspect, and we are there as well (Yaron Silberberg, Yaron Bromberg, Yoav Lahini, Eran and more ☺)

HBT used search-light mirrors to guide star light to two PMTs separated by up to 10 meters, to do this stellar light Intensity correlations (not interferometry) on Sirius (the brightest star) and got the Speckle size, and hence (since the distance is known) it's angular size – 0.0068"

Quantum bosonic interference effect – Fano's interpretation, but really not needed, and you can easily explain classically (no need for quantum), or easily turn to anti-bunching even though these are Bosons and not Fermions (as I have done in : PRA75_043804_(2007) "Spectral polarization and spectral phase control of time-energy entangled photons")

Show the same can be done for binary stars – the fringes will move.

The principle is obvious, however – we need to remember that we are dealing with correlations of not the intensities, but of the currents from the detectors (note – things are MUCH more problematic when you auto-correlate the same detector – don't do that. Even theoretically, we have an additional term of the shot noise of the current, as you can see in Mandel 452-464, and as I saw in my antibunching experiment (the added delta), but even worse – there are technical problems, like double-pulsing etc. So we derivate only for separate detectors),

So, assuming the light is stationary, and that the impulse-response of the detector at time *t* from detection at t_j is:

$$r(t - t_j)$$

Which is assumed to be zero for $t < t_j$ due to the mere argument of causality.

For example, it could be a current pulse that is *R* Amps high, for *Tr* (response time).

Let's define the total area (charge) of the each pulse:

$$Q = \int r(t)dt$$

We also assume that we integrate over time T that is much larger than the response time *TrI,* which enables integral and not summation, leading to (HW):

$$\langle J_1(t) J_2(t+\tau)\rangle = \eta_1\eta_1 Q^2 \langle I_1\rangle\langle I_1\rangle + \eta_1\eta_1 \iint \langle I_1(t) I_2(t+\tau+t'-t'')\rangle\, r(t')\, r(t'')\, dt' dt''$$

$$\langle \Delta J_1(t) \Delta J_2(t+\tau)\rangle = \eta_1\eta_1 \iint \langle \Delta I_1(t)\, \Delta I_2(t+\tau+t'-t'')\rangle\, r(t')\, r(t'')\, dt' dt''$$

With

$$J_{1,2}(t) = \langle J_{1,2}(t)\rangle + \Delta J_{1,2}(t)$$

$$I_{1,2}(t) = \langle I_{1,2}(t)\rangle + \Delta I_{1,2}(t)$$

Now we can assume two limits:

Fast detectors (untypical since 1nm=7500GHz !)

Then the detectors can be taken as delta functions during the integration over the intensity correlation, and naturally, then we get simply:

$$\langle \Delta J_1(t) \Delta J_2(t+\tau)\rangle = \eta_1\eta_1 Q^2 \langle \Delta I_1(t)\, \Delta I_2(t+\tau+t'-t'')\rangle$$

Slow detectors (typical)

Then the detectors can be taken as constants in the integration over the intensity correlation which is taken as delta:

$$\langle \Delta J_1(t)\Delta J_2(t+\tau)\rangle = \eta_1\eta_1 \langle \Delta I_1 \Delta I_2\rangle\, T_c \int r(t')\, r(t'+\tau)\, dt'$$

Where T_c is the correlation length of the Intensities.

HW: Assuming that the field has Gaussian statistics prove that for light polarized in the x direction :

$$\langle \Delta I_1(t)\Delta I_2(t+\tau)\rangle = \langle I_1\rangle\langle I_2\rangle \left|\gamma_{xx}^{(1)}(\boldsymbol{r_1},\boldsymbol{r_2},\tau)\right|^2$$

And for un-polarized we get similarly:

$$\langle \Delta I_1(t)\Delta I_2(t+\tau)\rangle = \frac{1}{2}\langle I_1\rangle\langle I_2\rangle \left|\gamma_{xx}^{(1)}(\boldsymbol{r_1},\boldsymbol{r_2},\tau)\right|^2$$

Using this:

$$\langle \Delta J_1(t)\Delta J_2(t+\tau)\rangle = \frac{1}{2}\eta_1\eta_1 \langle I_1\rangle\langle I_2\rangle \left|\gamma_{xx}^{(1)}(\boldsymbol{r_1},\boldsymbol{r_2},0)\right|^2 T_c \int r(t')\, r(t'+\tau)\, dt'$$

This is known as the Brown-Twiss effect, observed with Thermal light in 1956

For example, for square pulses at height *R*, length *Tr* , we'll get a triangle *2Tr* wide, and the height will be:

$$\eta_1\eta_2\langle I_1\rangle\langle I_2\rangle \left|\gamma_{xx}^{(1)}(\boldsymbol{r_1},\boldsymbol{r_2},0)\right|^2\ T_c T_r R^2$$

Let's compare that to the background, to get the SNR:

$$\eta_1\eta_2 Q^2\langle I_1\rangle\langle I_1\rangle = \eta_1\eta_1 (RT_r)^2\langle I_1\rangle\langle I_1\rangle$$

$$SNR = \left|\gamma_{xx}^{(1)}(\boldsymbol{r_1},\boldsymbol{r_2},0)\right|^2\ T_c/T_r$$

So we lose exactly by the factor of how slow our detectors are compared to the bandwidth of the light (filters won't help – they lower our signal by the same amount).

Narrabri Observatory in Australia – 2 6.5m reflectors, on rails of 188m in diameter – resolution of 0.00003" . With 65cm telescopes we have 1/100 Intensity = 1/10000 signal = 10 MAGNITUDES less !

Where classicality breaks ? when we assumed you can always divide light to two identical halves.. n,n-1:

Quantum Correlations

So we saw $g_c^{(2)}(0) \geq 1$ and $g_c^{(2)}(\tau) \leq g_c^{(2)}(0)$, and the methods that USE it (HBT)

Go back… we said coherent obeys Poissonian which means no added knowledge from detection of one to the other and so we expect…

$$g^{(2)}(0) = \frac{\langle\alpha|\hat{a}^\dagger\ \hat{a}^\dagger\ \hat{a}\ \hat{a}|\alpha\rangle}{\langle\alpha|\hat{a}^\dagger\ \hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger\ \hat{a}|\alpha\rangle} = \frac{|\alpha|^4}{|\alpha|^2|\alpha|^2} = 1$$

Indeed.

Remember that this means ZERO Variance – no fluctuations at all – coherent state is a PERFECT laser (no noise ?)

We already know from the Tutorial that states that have more noise, for example a Thermal State, which is the one that is in equilibrium with a blackbody and so is in Maxwell-Boltzmann distribution (page 26):

$$P_n = \left[\frac{1 - e^{-\hbar\omega/k_B T}}{e^{-\hbar\omega/2k_B T}}\right] e^{-\hbar\omega\left(n+\frac{1}{2}\right)/k_B T}$$

$$\rho_{Thermal} = \sum P_n\, |n\rangle\, \langle n|$$

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$$\bar{n} = \hbar\omega/k_B T \quad for\ \ \hbar\omega \gg k_B T\ ,\ \ which\ is\ in\ the\ Far\ IR\ and\ microwave\ in\ room\ temp$$

$$\bar{n} = k_B T/\hbar\omega \quad for\ \ k_B T \gg \hbar\omega$$

In the visible n~10^-40 !!

At the surface of the sun (~6000K) visible photons are $\bar{n}{\sim}10^{-2}$

So optics around us is governed by spontaneous emission, not induced – we NEVER came even close to n=1 before the invention of the laser !!

Anyway, Thermal of course has $g^{(2)}(0) = 2$

So the quantum expression for the variance allows 0 (g2=1) even in the presence of the MINIMAL noise of coherent state – shot noise, vacuum fluctuations…

But the quantum world is even MORE interesting, since it allows even LESS noise than NO noise:

Breaking the positive variance $g_c^{(2)}(0) < 1$ is called ***"Sub-Poissonian"*** (naturally, since it is less than the 1 of Poissonian).

We already know one state that should be sub-Poissonian – a Fock state:

$$g^{(2)}(0) = \frac{\langle n|\hat{a}^{\dagger}\,\hat{a}^{\dagger}\,\hat{a}\,\hat{a}\,|n\rangle}{\langle n|\hat{a}^{\dagger}\,\hat{a}\,|n\rangle\langle n|\hat{a}^{\dagger}\,\hat{a}\,|n\rangle} = \frac{n(n-1)}{n^2} = 1 - \frac{1}{n} < 1\ !$$

And even zero for SINGLE PHOTONS – until 10 years ago just showing such a state was a very big deal. HW: Show that for sub-Poissonian you need negative P representation

Even without reaching $g_c^{(2)}(0) < 1$, just reaching $g_c^{(2)}(\tau\ or\ \Delta) > g_c^{(2)}(0)$ is nonclassical, since it violates Cauchy-Shwarz, and this is called ***"anti-bunching"*** – having higher probability to detect two photons at different times or place, than to have two simultaneous detections at one of the places.

The first observation of anti-bunching is considered today the first true evidence for the quantization of the light field, i.e. the photons (NOT the photo-electric effect):

[H. J. Kimble, M. Dagenais, and L. Mandel, PRL 39, 691 (1977), "Photon Antibunching in Resonance Fluorescence"]

See also:

A. Kuhn, M. Hennrich, and G. Rempe, Phys. Rev. Lett. 89, 067901 (2002),

and Jeff Kimble's response which discusses g^(2):

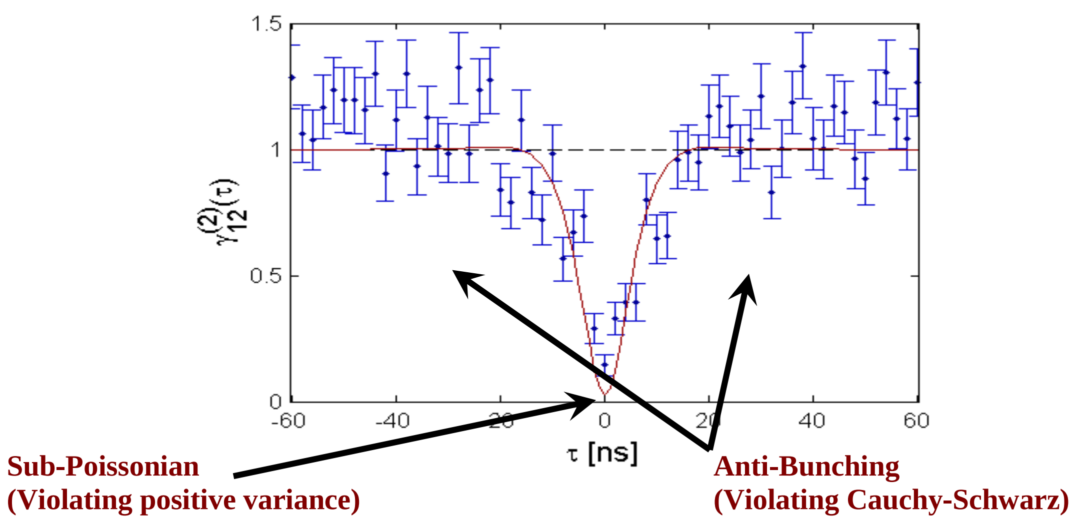
Phys. Rev. Lett. 90, 249801 (2003) Comment on "Deterministic Single-Photon Source for Distributed Quantum Networking"

And then his group's (Caltech) more solid results:

J. McKeever, *Andrea Boca, David Boozer, Russ Miller, J.R. Buck, Alex Kuzmich and Jeff H. Kimble, "*Deterministic Generation of Single Photons from One Atom Trapped in a Cavity" Science 26, 1992 (2004)

And from the same group – the cavity-QED demonstration of anti-bunching from resonance fluorescence:

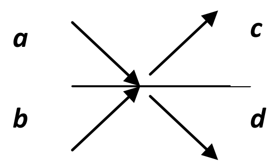
Science 319 1062 (2008) "A Photon Turnstile Dynamically Regulated by One Atom" B. Dayan. A. Parkins, T. Aoki, E. Ostby, K. H. Vahala and J. H. Kimble



The Homg-Ou_Mandel effect – first treatment – just a taste

C. K. Hong, Z. Y. Ou and L. Mandel, Phys. Rev. Lett. 59 2044 (1987) "Measurement of subPicosecond time intervals between Two Photons by Interference"

(I'll show the full temporal/spectral treatment of real two-photon interferometry in two weeks, when we'll know how to generate single photons or photon pairs. I'll also show how to turn it from Boson to Fermion, so Fano's interpretation for HBT is not really necessary, after all it is classical)



$$c = \frac{ia + b}{\sqrt{2}}, \qquad d = \frac{a + ib}{\sqrt{2}},$$

Using an input of a single photon in ***a*** and in ***b*** :

$$\varphi = |1,1\rangle$$

The number of clicks in ***c*** in each experiment:

$$\langle c^\dagger c\rangle = \frac{1}{2}\langle 1,1|(-ia^\dagger + b^\dagger)(ia + b)|1,1\rangle$$

$$= \frac{1}{2}\langle 1,1|a^\dagger a - ia^\dagger b + ib^\dagger a + b^\dagger b|1,1\rangle = 1$$

On average 1 click (1/2 of the cases)- as expected.

d will give the same of course.

Now look for coincidence detection:

$$\langle c^\dagger d^\dagger d\, c\rangle = \frac{1}{4}\langle 1,1|(-ia^\dagger + b^\dagger)(-ib^\dagger + a^\dagger)(ib + a)(ia + b)|1,1\rangle$$

$$= \frac{1}{4}\langle 1,1|(-ia^\dagger + b^\dagger)(-ib^\dagger + a^\dagger)(-ba + ibb + iaa + ab)|1,1\rangle$$

$$= 0 \quad \textit{since } a, b \textit{ commute, and } aa \textit{ or } bb \textit{ negate the state}$$

i.e. NEVER coincidences – you can see clearer why in the Schrodinger picture: the interference is destructive

$$2\,|1,1\rangle \rightarrow \;\; i|2,0\rangle + i\,|0,2\rangle + |1,1\rangle - \;|1,1\rangle$$

So we get 2 in ***c*** OR 2 in ***d***, and NEVER 1,1

Show it - the double detection at one of the detectors:

$$\langle c^{\dagger}c^{\dagger}c\,c\rangle = \frac{1}{4}\langle 1,1|\big(-ia^{\dagger}+b^{\dagger}\big)\big(-ia^{\dagger}+b^{\dagger}\big)(ia+b)(ia+b)\big|1,1\rangle$$

$$= \frac{1}{4}\langle 1,1|(H.C.)(-aa+iab+iba+bb)|1,1\rangle$$

$$= 1$$

Of course we'll get

$$\langle d^{\dagger}d^{\dagger}d\,d\rangle = 1$$

So after normalizing by the sum of all the possible events we get:

$$\frac{\langle c^{\dagger}c^{\dagger}c\,c\rangle}{\langle c^{\dagger}c^{\dagger}c\,c\rangle + \langle d^{\dagger}d^{\dagger}d\,d\rangle + \langle c^{\dagger}d^{\dagger}d\,c\rangle + \langle d^{\dagger}c^{\dagger}c\,d\rangle} = \frac{1}{2}$$

$$= \frac{\langle d^{\dagger}d^{\dagger}d\,d\rangle}{\langle c^{\dagger}c^{\dagger}c\,c\rangle + \langle d^{\dagger}d^{\dagger}d\,d\rangle + \langle c^{\dagger}d^{\dagger}d\,c\rangle + \langle d^{\dagger}c^{\dagger}c\,d\rangle} = \frac{1}{2}$$

i.e. in half of the cases we get 2 in c, and the other half is 2 in d

This is BUNCHING, and some would say it's because they are bosons, and this is only true in single mode. In more I can turn them to Fermions

How can we make PERFECT coincidences (anti-bunching) ?

Just REVERSE the process – put the output as input, and of course we'll get at the output the state $|1,1\rangle$.

So we take as input:

$$\varphi = \frac{|2,0\rangle + |0,2\rangle}{\sqrt{2}}$$

$$\langle c^{\dagger}d^{\dagger}d\,c\rangle = \frac{1}{8}\langle\varphi|\big(-ia^{\dagger}+b^{\dagger}\big)\big(-ib^{\dagger}+a^{\dagger}\big)(ib+a)(ia+b)\big|\varphi\rangle$$

$$= \frac{1}{8}[\langle 2,0| + \langle 0,2|]\left(\big(-ia^{\dagger}+b^{\dagger}\big)\big(-ib^{\dagger}+a^{\dagger}\big)(-ba+ibb+iaa+ab)\right)[|2,0+|0,2\rangle]$$

$$= 1 \quad since\ each\ side\ gives\ 2\sqrt{2}$$

We must also sum the probability for

$$\langle d^\dagger c^\dagger c\, d\rangle = 1$$

As well. On the other hand:

$$\langle c^\dagger c^\dagger cc\rangle = \frac{1}{4}\langle\varphi\big|\left(-ia^\dagger + b^\dagger\right)\left(-ia^\dagger + b^\dagger\right)(ia + b)(ia + b)\big|\varphi\rangle$$

$$= \frac{1}{8}[\langle 0,2| + \langle 0,2|]\left(\left(-ia^\dagger + b^\dagger\right)\left(-ib^\dagger + a^\dagger\right)(-aa + iab + iba + bb)\right)[|0,2\rangle + |0,2\rangle]$$

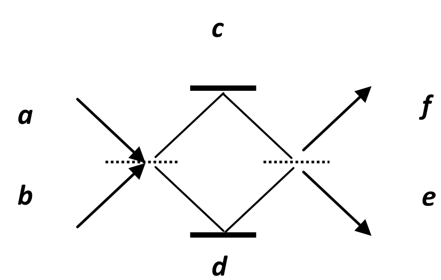
$$= 0 \;\; since \;\; aa \; and \; bb \; will \; interfere \; to \; the \; vacuum \; state \; destructively,$$

$$and \; the \; ab \; and \; ba \; just \; negate \; the \; state$$

Since these two options (2 in *c* or 2 in *d*) give zero, after normalizing we see that indeed we always a coincidence.

Later I'll show how you can use it to generate two single photons in different modes by interfering down conversion of photon pairs from two crystals.

Next, let's look at a Mach-Zender interferometer:



$$c = \frac{ia + b}{\sqrt{2}}, \qquad d = \frac{a + ib}{\sqrt{2}},$$

For BALANCED Mach-Zender:

$$e = \frac{ia + b + ia - b}{2} = ia\,, \qquad f = \frac{a + ib - a + ib}{2} = ib\,,$$

Direct mapping – whatever comes in *a (b)* gets out in *e(f)*

Now we add a phase (=delay assuming single mode):

$$e = \frac{ia + b + ie^{i\theta}(a + ib)}{2}\,, \qquad f = \frac{e^{i\theta}(a + ib) - a + ib}{2},$$

$$e = \frac{ia\left(1 + e^{i\theta}\right) + b\left(1 - e^{i\theta}\right)}{2} = ie^{i\frac{\theta}{2}}\left[a\cos\frac{\theta}{2} - b\sin\frac{\theta}{2}\right],$$

$$f = \frac{a\left(e^{i\theta} - 1\right) + ib\left(1 + e^{i\theta}\right)}{2} = ie^{i\frac{\theta}{2}}\left[b\cos\frac{\theta}{2} + a\sin\frac{\theta}{2}\right]$$

Finally – a Mach-Zender is the basis for Elitzur & Vaidman's Bomb test:

Avshalom C. Elitzur and Lev Vaidman, "Quantum mechanical interaction-free measurements". *Foundations of Physics* **23** (1993), 987-97

= "Interaction-Free Measurement"

All cavity-QED experiments (mine included) begin with such "Interaction-free measurement", only that is close to 100% efficient, with nearly 0% chances for "activating the bomb", but nobody mentioned it until Jakob Reichel's work that got a lot of "mileage" from it, including demonstrating Zeno:

Nature 475 210 (2011) "Measurement of the internal state of a single atom without energy exchange" , Jürgen Volz, Roger Gehr, Guilhem Dubois, Jérôme Estève & Jakob Reichel