$$L = \sum_{i=1,2}\left[\frac{L_0\dot Q_i^2 + L_2\dot Q_i^4}{2} - \frac{Q_i^2}{2C} - \frac{(Q_{di}-Q_i)^2}{2C_d}\right] - \frac{(Q_1-Q_2)^2}{2C_c}$$

$$\begin{cases} L_0\ddot Q_1 + 6L_2\dot Q_1^2\ddot Q_1 + \dfrac{Q_1}{C} + \dfrac{Q_1-Q_2}{C_c} = -\dfrac{Q_{di}-Q_1}{C_d} \\ L_0\ddot Q_2 + 6L_2\dot Q_2^2\ddot Q_2 + \dfrac{Q_2}{C} - \dfrac{Q_1-Q_2}{C_c} = -\dfrac{Q_{di}-Q_2}{C_d} \end{cases}$$

$$\begin{cases} \ddot Q_1 + \omega_0^2 Q_1 - KQ_2 + \beta\dot Q_1^2\ddot Q_1 = \varepsilon_1\cos(\omega_p t) \\ \ddot Q_2 + \omega_0^2 Q_2 - KQ_1 + \beta\dot Q_2^2\ddot Q_2 = \varepsilon_2\cos(\omega_p t) \end{cases}$$

drivers

$$\boxed{\omega_0^2 = \frac{1}{CL_0} + \frac{1}{C_cL_0}, \quad K = \frac{1}{L_0C_c}, \quad \beta = \frac{6L_2}{L_0}}$$

New Lagrangian:

$$L = \frac{\dot Q_1^2}{2} \mp \frac{\omega_0^2 Q_1^2}{2} + \frac{\beta\dot Q_1^4}{12} + \frac{\dot Q_2^2}{2} \mp \frac{\omega_0^2 Q_2^2}{2} + \frac{\beta\dot Q_2^4}{12} \mp KQ_1Q_2$$
$$\mp \varepsilon_1 Q_1\cos(\omega_p t) \mp \varepsilon_2 Q_2\cos(\omega_p t)$$

Next, assume $\varepsilon_2 = 0$ and transform to normal coordinates

$$Q_1 = x_1 + x_2, \quad Q_2 = x_1 - x_2$$

$$\boxed{L = \frac{\dot x_1^2}{2} + \frac{\omega_1^2 x_1^2}{2} + \frac{\dot x_2^2}{2} + \frac{\omega_2^2 x_2^2}{2} + \frac{\beta}{12}\left[(\dot x_1+\dot x_2)^4 + (\dot x_1-\dot x_2)^4\right] + \varepsilon_1(x_1+x_2)\cos(\omega_p t)}$$

$$L = \frac{\dot x_1^2}{2} + \frac{\omega_1^2 x_1^2}{2} + \frac{\dot x_2^2}{2} + \frac{\omega_2^2 x_2^2}{2} + \frac{\beta}{6}\left(\dot x_1^4 + \dot x_2^4 + 6\dot x_1^2\dot x_2^2\right)$$
$$+ \varepsilon_1(x_1+x_2)\cos(\omega_p t)$$

$\omega_1^2 = \omega_0^2 \pm K$

$$\begin{cases} \ddot x_1 + \omega_1^2 x_1 + \dfrac{\beta}{6}\left[4(\dot x_1^3)^{\cdot} + 12(\dot x_1\dot x_2^2)^{\cdot}\right] = \varepsilon\cos(\omega_p t) \\ \ddot x_2 + \omega_2^2 x_2 + \dfrac{\beta}{6}\left[4(\dot x_2^3)^{\cdot} + 12(\dot x_2\dot x_1^2)^{\cdot}\right] = \varepsilon\cos(\omega_p t) \end{cases}$$

$$\begin{aligned} \ddot x_1 + \omega_1^2 x_1 + 2\beta\left[\dot x_1^2\ddot x_1 + \boxed{\ddot x_1\dot x_2^2 + 2\dot x_1\dot x_2\ddot x_2}\right] &= \varepsilon\cos(\omega_p t) \\ \ddot x_2 + \omega_2^2 x_2 + 2\beta\left[\dot x_2^2\ddot x_2 + \boxed{\ddot x_2\dot x_1^2 + 2\dot x_1\dot x_2\ddot x_1}\right] &= \varepsilon\cos(\omega_p t) \end{aligned}$$

parametric resonance comes from these terms

Seek solutions

$$x_{1,2} = x^0_{1,2} + \tilde{x}_{1,2}, \qquad x^0_{1,2} = \frac{\varepsilon}{\omega_{1,2}^2-\omega_p^2}\cos(\omega_p t) = a^0_{1,2}\cos(\omega_p t)$$

$$\tilde{x}_{1,2} = a_{1,2}\cos(\tilde{\omega}_{1,2} t)$$

where $\tilde{\omega}_{1,2} \approx \omega_{1,2}$

$$\begin{cases} \ddot{\tilde{x}}_1 + \omega_1^2\tilde{x}_1 + 2\beta\left[(\dot{x}_1^0+\dot{\tilde{x}}_1)^2(\ddot{x}_1^0+\ddot{\tilde{x}}_1) + (\ddot{x}_1^0+\ddot{\tilde{x}}_1)(\dot{x}_2^0+\dot{\tilde{x}}_2)^2 + 2(\dot{x}_1^0+\dot{\tilde{x}}_1)(\dot{x}_2^0+\dot{\tilde{x}}_2)(\ddot{x}_2^0+\ddot{\tilde{x}}_2)\right] = 0 \\ \ddot{\tilde{x}}_2 + \omega_2^2\tilde{x}_2 + 2\beta\left[(\dot{x}_2^0+\dot{\tilde{x}}_2)^2(\ddot{x}_2^0+\ddot{\tilde{x}}_2) + (\ddot{x}_2^0+\ddot{\tilde{x}}_2)(\dot{x}_1^0+\dot{\tilde{x}}_1)^2 + 2(\dot{x}_1^0+\dot{\tilde{x}}_1)(\dot{x}_2^0+\dot{\tilde{x}}_2)(\ddot{x}_1^0+\ddot{\tilde{x}}_1)\right] = 0 \end{cases}$$

Leave resonant contributions only

$$\boxed{\dot{\tilde{x}}_1^2\ddot{\tilde{x}}_1 \longrightarrow -\frac{3}{4}\omega_1^4 a_1^3\cos(\tilde{\omega}_1 t)}$$

$$(\ddot{x}_1^0+\ddot{\tilde{x}}_1)(\dot{x}_2^0+\dot{\tilde{x}}_2)^2 \longrightarrow 2\ddot{x}_1^0\dot{x}_2^0\dot{\tilde{x}}_2 \longrightarrow -2\omega_p^3\omega_2 a_1^0 a_2^0 a_2\cos(\omega_p t)\sin(\omega_p t)\sin(\tilde{\omega}_2 t)$$

$$\to -\frac{1}{2}\omega_p^3\omega_2 a_1^0 a_2^0 a_2\cos\left[(2\omega_p-\tilde{\omega}_2)t\right]$$

$$2(\dot{x}_1^0+\dot{\tilde{x}}_1)(\dot{x}_2^0+\dot{\tilde{x}}_2)(\ddot{x}_2^0+\ddot{\tilde{x}}_2) \longrightarrow 2\dot{x}_1^0\ddot{x}_2^0\dot{\tilde{x}}_2 + 2\dot{x}_1^0\dot{x}_2^0\ddot{\tilde{x}}_2$$

$$\longrightarrow -2\omega_p^3\omega_2 a_1^0 a_2^0 a_2\sin(\omega_p t)\cos(\omega_p t)\sin(\tilde{\omega}_2 t) - 2\omega_p^2\omega_2^2 a_1^0 a_2^0 a_2\sin^2(\omega_p t)\cos(\tilde{\omega}_2 t)$$

$$\to \frac{1}{2}\omega_p^2\omega_2 a_1^0 a_2^0 a_2(\omega_p-\omega_2)\cos\left[(2\omega_p-\tilde{\omega}_2)\right]$$

Then in the resonance approximation our system becomes

$$(\omega_1^2-\tilde{\omega}_1^2)a_1\cos(\tilde{\omega}_1 t) - \frac{3\beta}{2}\omega_1^4 a_1^3\cos(\tilde{\omega}_1 t) = \beta\omega_p^2\omega_2^2 a_1^0 a_2^0 a_2\cos\left[(2\omega_p-\tilde{\omega}_2)t\right]$$

$$(\omega_2^2-\tilde{\omega}_2^2)a_2\cos(\tilde{\omega}_2 t) - \frac{3\beta}{2}\omega_2^4 a_2^3\cos(\tilde{\omega}_2 t) = \beta\omega_p^2\omega_1^2 a_1^0 a_2^0 a_1\cos\left[(2\omega_p-\tilde{\omega}_1)t\right]$$

Next we set $\underbrace{\tilde{\omega}_1 = 2\omega_p-\tilde{\omega}_2 \text{ and } \tilde{\omega}_2 = 2\omega_p-\tilde{\omega}_1}_{\text{the same}}$

$$\boxed{\begin{aligned} (\omega_1^2-\tilde{\omega}_1^2)a_1 - \cancel{\frac{3\beta}{2}\omega_1^4 a_1^3} &= r_1 a_2 \\ (\omega_2^2-\tilde{\omega}_2^2)a_2 - \cancel{\frac{3\beta}{2}\omega_2^4 a_2^3} &= r_2 a_1 \end{aligned}} \longleftarrow \boxed{r_{1,2} = \beta\omega_p^2\omega_{1,2}^2 a_1^0 a_2^0}$$

Neglect nonlinear frequency shifts for small $a_{1,2}$

$$\begin{cases} (\omega_1^2-\tilde{\omega}_1^2)\, a_1 = r_1 a_2 \\ (\omega_2^2-\tilde{\omega}_2^2)\, a_2 = r_2 a_1 \end{cases} \rightarrow (\omega_2^2-\tilde{\omega}_2^2)(\omega_1^2-\tilde{\omega}_1^2) = r_1 r_2$$

$$\underbrace{(\omega_2+\tilde{\omega}_2)}_{\approx 2\omega_2}(\omega_2-\tilde{\omega}_2)\underbrace{(\omega_1+\tilde{\omega}_1)}_{\approx 2\omega_1}(\omega_1-\tilde{\omega}_1) = r_1 r_2$$

$$(\omega_2-\tilde{\omega}_2)(\omega_1-\tilde{\omega}_1) = \frac{r_1 r_2}{2\omega_1\omega_2} = \gamma^2$$

Let $\tilde{\omega}_1 = \omega_1 + \delta \rightarrow \tilde{\omega}_2 = 2\omega_p - \tilde{\omega}_1 = 2\omega_p - \omega_1 - \delta$

$$-(\underbrace{\omega_2+\omega_1-2\omega_p}_{\Delta} + \delta)\,\delta = \gamma^2$$

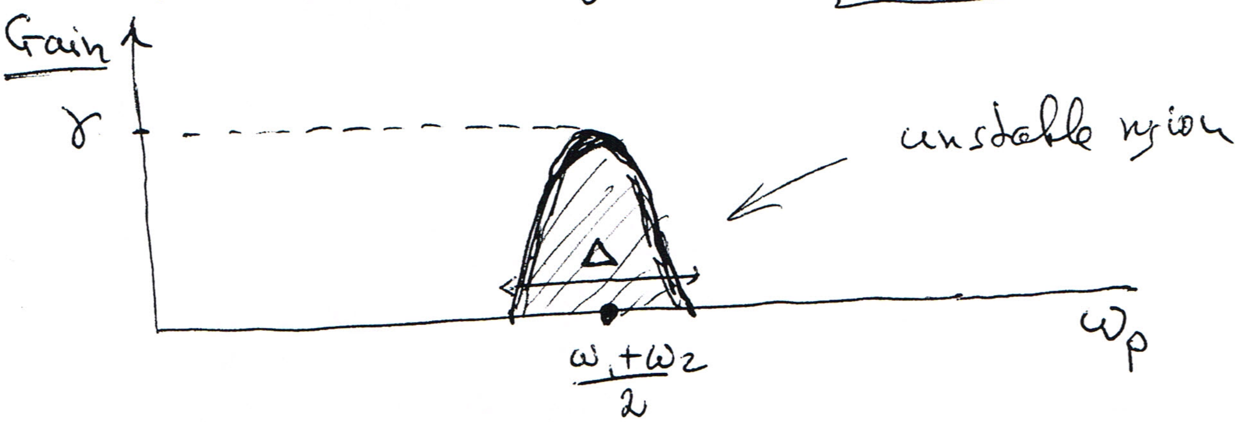
Δ is the phase shift

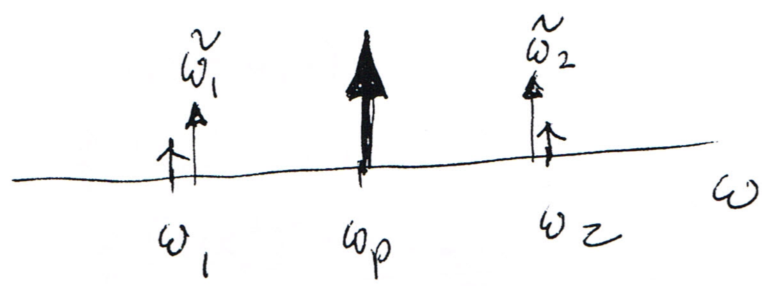
$\Delta = \omega_1+\omega_2-2\omega_p$

$$\delta^2 + \Delta\delta + \gamma^2 = 0$$

$$\delta_\pm = -\frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} - \gamma^2}$$

We have instability for $\boxed{|\Delta| < 2\gamma}$





Nonlinear shifts ? Pump depletion ?