Original Lagrangian

$$L = \sum_{i=1,2} \frac{L_0 \dot{\varphi}_i^2 + L_2 \dot{\varphi}_i^4}{2} - \frac{\varphi_i^2}{2C} - \frac{(\varphi_{d_i} - \varphi_i)^2}{2C_d} - \frac{(\varphi_1 - \varphi_2)^2}{2C_c}$$

Evolution equations

$$\begin{cases} L_0 \ddot{\varphi}_1 + 6 L_2 \dot{\varphi}_1^2 \ddot{\varphi}_1 + \frac{\varphi_1}{C} + \frac{\varphi_1 - \varphi_2}{C_c} = \frac{\varphi_{d1} - \varphi_1}{C_d} \\ L_0 \ddot{\varphi}_2 + 6 L_2 \dot{\varphi}_2^2 \ddot{\varphi}_2 + \frac{\varphi_2}{C} - \frac{\varphi_1 - \varphi_2}{C_c} = \frac{\varphi_{d2} - \varphi_2}{C_d} \end{cases}$$

$$\begin{cases} \ddot{\varphi}_1 + \omega_0^2 \varphi_1 - k\varphi_2 + \beta \dot{\varphi}_1^2 \ddot{\varphi}_1 = \varepsilon_1 \cos(\omega_p t) \\ \ddot{\varphi}_2 + \omega_0^2 \varphi_2 - k\varphi_1 + \beta \dot{\varphi}_2^2 \ddot{\varphi}_2 = \varepsilon_2 \cos(\omega_p t) \end{cases}$$

$$\boxed{\omega_0^2 = \frac{1}{a_0} + \frac{1}{C_c L_0}, \quad k = \frac{1}{L_0 C_c}, \quad \beta = \frac{6 L_2}{L_0}}$$

Simplified Lagrangian

$$L = \frac{1}{2}\left(\dot{\varphi}_1^2 - \omega_0^2 \varphi_1 + \dot{\varphi}_2^2 - \omega_0^2 \varphi_2 + 2k\varphi_1\varphi_2\right) + \frac{\beta}{12}\left(\dot{\varphi}_1^4 + \dot{\varphi}_2^4\right)$$
$$+ \varepsilon_1 \varphi_1 \cos(\omega_p t) + \varepsilon_2 \varphi_2 \cos(\omega_p t)$$

Assume $\varepsilon_2 = 0$ and transform to normal coordinates

$$\varphi_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad \varphi_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

New Lagrangian

$$L = \frac{1}{2}\left(\dot{x}_1^2 - \omega_1^2 x_1 + \dot{x}_2^2 - \omega_2^2 x_2\right) + \frac{\beta}{24}\left(\dot{x}_1^4 + \dot{x}_2^4 + 6\dot{x}_1^2 \dot{x}_2^2\right) + \varepsilon (x_1 + x_2) \cos(\omega_p t)$$

$$\omega_{1,2}^2 = \omega_0^2 \pm k, \quad \varepsilon = \varepsilon_1 / \sqrt{2}$$

$\theta_p = \omega_p t$

Evolution equations

$$\ddot{x}_1 + \omega_1^2 x_1 + \frac{\beta}{24}\left(4(\dot{x}_1^3)^{\cdot} + 12(\dot{x}_1 \dot{x}_2^2)^{\cdot}\right) = \varepsilon \cos \theta_p$$
$$\ddot{x}_2 + \omega_2^2 x_2 + \frac{\beta}{24}\left(4(\dot{x}_2^3)^{\cdot} + 12(\dot{x}_2 \dot{x}_1^2)^{\cdot}\right) = \varepsilon \cos \theta_p$$

Seek solutions

$$x_{1,2} = x_{1,2}^0 + y_{1,2}, \quad \boxed{\begin{aligned} x_{1,2}^0 &= \frac{\varepsilon}{\omega_{1,2}^2 - \omega_p^2} \cos \theta_p \\ y_{1,2} &= a_{1,2} \cos \theta_{1,2} \\ \dot{\theta}_{1,2} &= \nu_{1,2} \simeq \omega_{1,2} \end{aligned}}$$

$= a_{1,2}^0 \cos \theta_p$

$$\begin{cases} (\omega_1^2-\nu_1^2)\,a_1\cos\Theta_1 + \frac{\beta}{6}\left[(\dot{x}_1^3)^\circ + 3(\dot{x}_1\dot{x}_2^2)^\circ\right] = 0 \\ (\omega_2^2-\nu_2^2)\,a_2\cos\Theta_2 + \frac{\beta}{6}\left[(\dot{x}_2^3)^\circ + 3(\dot{x}_1\dot{x}_2^2)^\circ\right] = 0 \end{cases}$$

$\nu_1 \simeq \omega_1$

$$\dot{x}_1^3 = -(\omega_p a_1^\circ \sin\Theta_p + \nu_1 a_1 \sin\Theta_1)^3 =$$

$$= -(\omega_p^3 a_1^{\circ 3}\sin^3\Theta_p + 3\omega_p^2 a_1^{\circ 2}\nu_1 a_1 \sin^2\Theta_p\sin\Theta_1 + 3\omega_p a_1^\circ \nu_1^2 a_1^2\sin^2\Theta_1 + \nu_1^3 a_1^3\sin^3\Theta_1)$$

We seek phases Θ_1 and $2\Theta_p-\Theta_2$ in the first evolution equation. There are two terms in $\dot{x}_1^3$ yielding Θ_1 contribution

$$3\sin^2\Theta_p\sin\Theta_1 = 3\,\frac{1-\cancel{\cos 2\Theta_p}}{2}\sin\Theta_1 \longrightarrow \frac{3}{2}\sin\Theta_1$$

$$\sin^3\Theta_1 = \sin\Theta_1\,\frac{1-\cos 2\Theta_1}{2} \Rightarrow \frac{1}{2}\sin\Theta_1 - \frac{1}{2}\sin\Theta_1\cos 2\Theta_1 =$$

$$= \frac{1}{2}\sin\Theta_1 - \frac{1}{4}(\sin\Theta_1 + \cancel{\sin 3\Theta_1}) \rightarrow \frac{1}{4}\sin\Theta_1$$

Thus

$$\boxed{\dot{x}_1^3 \Rightarrow -\left[\frac{3}{2}\omega_p^2\omega_1 a_1^{\circ 2} a_1\sin\Theta_1 + \frac{1}{4}\omega_1^3 a_1^3\sin\Theta_1\right]}$$

Next

$$(\dot{x}_1\dot{x}_2^2) = -(\omega_p a_1^\circ\sin\Theta_p + \underset{\approx\nu_1}{\omega_1} a_1\sin\Theta_1)(\omega_p a_2^\circ\sin\Theta_p + \underset{\approx\nu_2}{\omega_2} a_2\sin\Theta_2)^2$$

$$= -(\omega_p a_1^\circ\sin\Theta_p + \omega_1 a_1\sin\Theta_1)(\omega_p^2 a_2^{\circ 2}\sin^2\Theta_p + \omega_2^2 a_2^2\sin^2\Theta_2 + 2\omega_p\omega_2 a_2 a_2^\circ \sin\Theta_p\sin\Theta_2)$$

Here we have two contributing terms

$$\sin^2\Theta_p\sin\Theta_2 = \frac{1-\cos 2\Theta_p}{2}\sin\Theta_2 \Rightarrow -\frac{1}{4}\sin(2\Theta_p-\Theta_2)$$

$$\sin\Theta_1\sin^2\Theta_p = \sin\Theta_1\,\frac{1-\cancel{\cos 2\Theta_p}}{2} \Rightarrow \frac{1}{2}\sin\Theta_1$$

Thus

$$\boxed{3\dot{x}_1\dot{x}_2^2 \Rightarrow -\frac{3}{2}\omega_1^2\omega_p^2 a_2^{\circ 2} a_1\sin\Theta_1 + \frac{3}{2}\omega_1\omega_p^2\omega_2 a_1^\circ \overset{a_2}{a_2^\circ}\sin(2\Theta_p-\Theta_2)}$$

Now we assume $\Theta_1 = 2\Theta_p - \Theta_2$ and ~~therefore~~ therefore

$$\omega_p = \frac{\nu_1+\nu_2}{2} \approx \frac{\omega_1+\omega_2}{2}$$

Then the first evolution equation yields

$$(\omega_1^2-\nu_1^2)\,a_1 - \frac{\beta}{6}\left(\frac{3}{2}\omega_p^2\omega_1 a_1^{\circ 2}a_1 + \frac{1}{4}\omega_1^4 a_1^3\right)$$

$$+ \frac{\beta}{4}\left(-\omega_1^2\omega_p^2 a_2^{\circ 2}a_1 + \omega_1\omega_p^2\omega_2 a_1^\circ a_2^\circ a_2\right) = 0$$

or
$$\left[\omega_1^2-\nu_1^2 - \frac{\beta}{4}\omega_p^2\omega_1^2\left(a_1^{02}+a_2^{02}\right)\right]a_1 - \boxed{\frac{\beta}{24}\omega_1^4 a_1^3} + \frac{\beta}{4}\omega_1\omega_2\omega_p^2 a_1^0 a_2^0\, a_2 = 0$$
similarly
$$\left[\omega_2^2-\nu_2^2 - \frac{\beta}{4}\omega_p^2\omega_2^2\left(a_1^{02}+a_2^{02}\right)\right]a_2 - \boxed{\frac{\beta}{24}\omega_2^4 a_2^4} + \frac{\beta}{4}\omega_1\omega_2\omega_p^2 a_1^0 a_2^0\, a_1 = 0$$

$\approx 2\omega_1(\omega_1-\nu_1)$

neglect the nonlinearity

$$\begin{bmatrix} (\omega_1^2-\nu_1^2) - \frac{\beta}{4}\omega_p^2\omega_1^2(a_1^{02}+a_2^{02}) & \frac{\beta}{4}\omega_1\omega_2\omega_p^2 a_1^0 a_2^0 \\ \frac{\beta}{4}\omega_1\omega_2\omega_p^2 a_1^0 a_2^0 & \omega_2^2-\nu_2^2 - \frac{\beta}{4}\omega_p^2\omega_2^2(a_1^{02}+a_2^{02}) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

$$\left[2\omega_1(\omega_1-\nu_1) - \frac{\beta}{4}\omega_p^2\omega_1^2(a_1^{02}+a_2^{02})\right]\left[2\omega_2(\omega_2-\nu_2) - \frac{\beta}{4}\omega_p^2\omega_2^2(a_1^{02}+a_2^{02})\right]$$
$$- \frac{\beta^2}{16}\omega_1^2\omega_2^2\omega_p^4 a_1^{02}a_2^{02} = 0$$
$$\left[\omega_1-\nu_1 - \frac{\beta}{8}\omega_p^2\omega_1(a_1^{02}+a_2^{02})\right]\left[\omega_2-\nu_2 - \frac{\beta}{8}\omega_p^2\omega_2(a_1^{02}+a_2^{02})\right] - \frac{\beta^2}{64}\omega_1\omega_2\omega_p^4 a_1^{02}a_2^{02} = 0$$
Here ω_p^2 appears in β terms and thus can be replaced by $(\omega_1+\omega_2)/2$.

Next, we define $\delta = \nu_1 - \omega_1 + \frac{\beta}{8}\omega_p^2\omega_1(a_1^{02}+a_2^{02})$
Consequently (recall $\nu_1+\nu_2 = 2\omega_p$)
$$\omega_2 - \nu_2 - \frac{\beta}{8}\omega_p^2\omega_2(a_1^{02}+a_2^{02}) = \underbrace{-2\omega_p+\omega_1+\omega_2}_{\Delta} + \delta - \frac{\beta}{8}\omega_p^2\omega_1(a_1^{02}+a_2^{02}) - \frac{\beta}{8}\omega_p^2\omega_2(a_1^{02}+a_2^{02})$$
Then
$$+\delta\left[\delta + \underbrace{\Delta - \frac{\beta}{8}\omega_p^2(\omega_1+\omega_2)(a_1^{02}+a_2^{02})}_{\Delta'}\right] + \underbrace{\frac{\beta^2}{64}\omega_1\omega_2\omega_p^4 a_1^{02}a_2^{02}}_{r^2} = 0$$
$$\delta^2 + \Delta'\delta + \underbrace{\left(\frac{\beta^2}{64}\omega_1\omega_2\omega_p^4 a_1^{02}a_2^{02}\right)}_{r^2} = 0$$
$$\delta_{1,2} = -\frac{\Delta'}{2} \pm \sqrt{\frac{\Delta'^2}{4} - r^2}$$
Thus, we have instability for
$$|\Delta'| < 2r$$

This yields

$$-\frac{\beta}{4}\sqrt{\omega_1\omega_2}\,\omega_p^2|q_1^0 q_2^0| < \Delta - \frac{\beta}{8}\omega_p^2(\omega_1+\omega_2)(q_1^{02}+q_2^{02}) < \frac{\beta}{4}\sqrt{\omega_1\omega_2}\,\omega_p^2|q_1^0 q_2^0|$$

$$\underbrace{\left[(\omega_1+\omega_2)(q_1^{02}+q_2^{02}) - 2\sqrt{\omega_1\omega_2}|q_1^0 q_2^0|\right]}_{\text{always positive}} < \frac{8\Delta}{\beta\omega_p^2} < \left[(\omega_1+\omega_2)(q_1^{02}+q_2^{02}) + 2\sqrt{\omega_1\omega_2}|q_1^0 q_2^0|\right]$$

Thus, the unstable region is centered around

$$\Delta_0 = \frac{\beta\omega_p^2}{8}(\omega_1+\omega_2)(q_1^{02}+q_2^{02})$$

and has the width of

$$D = \frac{1}{2}\beta\omega_p^2\sqrt{\omega_1\omega_2}\,|q_1^0 q_2^0|$$